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INVERSION OF LAGRANGE'S THEOREM FOR A RIGID BODY WITH A CAVITY CONTAINING AN IDEAL LIQUID*

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A linear stability analysis is presented for the equilibrium state of a rigid body with a cavity completely filled with an ideal incompressible liquid possessing surface tension. The Lyapunov technique is used to show that the system is unstable if the second variation of the potential energy is allowed to take negative values. An estimate is derived which guarantees exponential growth of the mean-square deviations from equilibrium of the particles of the body and the liquid. The analysis employs a Lyapunov functional first defined in /1/.

1. *Lyapunov functional.* A criterion was developed in /1/ for the equilibrium state of a rigid body, containing a cavity partly or completely filled with an ideal incompressible liquid possessing surface tension, to be unstable. The proof made use of the following functional (/1/, p.179):

$$V = -(T + \Pi)\left\{\sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}} q_j + \sum_{i=1}^{3} \rho \int_{\tau} \frac{\partial T^\circ}{\partial u_i} \Delta x_i \, d\tau\right\} \tag{1.1}$$

$$\Bigl(T = T_1 + T_2, \quad \Pi = -U_1 - \rho \int_{\tau} U_2 \, d\tau + \Pi_2', \quad T_1 = T_1(q_j, q_j^{\cdot}),$$

$$T_2 = \rho \int_{\tau} T^\circ d\tau, \quad T^\circ = T^\circ(q_j, q_j^{\cdot}, x_i, u_i), \quad L = T - \Pi\Bigr)$$

Here T and Π are the kinetic and potential energy of the "body plus liquid" system, T_1 and $U_1(q_j)$ are the kinetic energy of the rigid body and the force function of the applied active forces, $q_j, q_j^{\cdot}$ $(j = 1, \ldots, n \leqslant 6)$ are the generalized coordinates and velocities of the rigid body, T_2 and $U_2(q_j, x_i)$ are the kinetic energy of the liquid and the force function of the mass forces acting on it, T° is the kinetic energy density of the liquid, x_i $(i = 1, 2, 3)$ are the Cartesian coordinates of the liquid particles in a reference frame rigidly attached to the body, $u_i = dx_i/dt$ are the relative velocities of the liquid particles, $\Delta x_i = x_i - x_{i0}$ are the displacements of the liquid particles relative to their equilibrium position x_{i0}, τ is the region of $x_1x_2x_3$ space occupied by the liquid, $\partial T^\circ/\partial u_i = v_i$ are the projections of the absolute velocity of the liquid, and Π_2' is the potential energy of the surface tension forces. It is assumed that at equilibrium $q_j = 0$ $(j = 1, \ldots, n)$.

It will be convenient to rewrite the functional in braces in (1.1) as follows:

$$W = \sum_{j=1}^{n} \frac{\partial T}{\partial q_j^{\cdot}} q_j + \sum_{i=1}^{3} \int_{\tau} \rho v_i \Delta x_i \, d\tau \tag{1.2}$$

Our attention will be confined henceforth to the first (linear) approximation. We will first show that to a first approximation W can be expressed as

$$W = \int \rho \mathbf{v} \cdot \boldsymbol{\xi} \, d\tau \tag{1.3}$$

Throughout this paper the integration is performed over the region of space $\tau_1 \cup \tau$ occupied by the body (τ_1) and the liquid (τ); $\boldsymbol{\xi}$ is the vector of displacement from equilibrium of the particles of the body or the liquid.

Indeed, the derivative of the kinetic energy of the "body plus liquid" system is

$$\frac{\partial T}{\partial q_j^{\cdot}} = \int \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_j^{\cdot}} \, d\tau, \quad T = \frac{1}{2} \int \rho \mathbf{v}^2 d\tau \tag{1.4}$$

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The radius-vector of a particle of the body relative to the origin of the inertial reference frame is

$$\mathbf{r} = \mathbf{r}(q_j) \tag{1.5}$$

and that of a liquid particle is

$$\mathbf{r} = \mathbf{r}(q_j, x_i) \tag{1.6}$$

so that the absolute velocity vector of the body is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \sum_{j=1}^{n} \frac{\partial \mathbf{r}}{\partial q_j} q_j^{\cdot}$$

and that of the liquid

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \sum_{j=1}^{n} \frac{\partial \mathbf{r}}{\partial q_j} q_j^{\cdot} + \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial x_i} u_i$$

Consequently, for points of both body and liquid,

$$\partial \mathbf{v}/\partial q_j^{\cdot} = \partial \mathbf{r}/\partial q_j$$

whence we obtain from (1.4), to a first approximation,

$$\frac{\partial T}{\partial q_j^{\cdot}} = \int \rho \mathbf{v} \cdot \left(\frac{\partial \mathbf{r}}{\partial q_j}\right)_0 d\tau \tag{1.7}$$

(the subscript zero means that the function is evaluated at equilibrium). On the other hand, it follows from (1.5), (1.6) that in the linear approximation the displacement vector $\xi \equiv \mathbf{r} - \mathbf{r}_0$ relative to the equilibrium position $\mathbf{r}_0$ of a particle of the body is

$$\xi = \sum_{j=1}^{n} \left(\frac{\partial \mathbf{r}}{\partial q_j}\right)_0 q_j \tag{1.8}$$

and that of a particle of the liquid is

$$\xi = \sum_{j=1}^{n} \left(\frac{\partial \mathbf{r}}{\partial q_j}\right)_0 q_j + \sum_{i=1}^{3} \left(\frac{\partial \mathbf{r}}{\partial x_i}\right)_0 \Delta x_i \tag{1.9}$$

But if ξ_s is the displacement vector of a point of the system, considered as a single rigid body, then

$$\xi_s = \sum_{j=1}^{n} \left(\frac{\partial \mathbf{r}}{\partial q_j}\right)_0 q_j$$

where $\xi = \xi_s$ for the rigid body and $\xi = \xi_s + \Delta \mathbf{x}$ for the liquid. Hence (1.7) implies a representation for the first term on the right of (1.2):

$$\sum_{j=1}^{n} q_j \frac{\partial T}{\partial q_j^{\cdot}} = \int \rho \mathbf{v} \cdot \xi_s \, d\tau$$

which immediately implies the reduction of the functional (1.2) to the form (1.3).

2. Estimate for the growth of perturbations. Suppose that the "body plus liquid" system is in its equilibrium state (at rest), but that the potential energy Π in this state does not have a minimum. We shall deduce from the equations of the first (linear) approximation that the system is unstable and give an estimate of the rate at which the perturbations increase.

Let the quantities $u_i, \Delta x_i, q_j, q_j^{\cdot}, v_i, \xi_i$ satisfy the linearized governing equations and boundary conditions, which are readily derived from those of /1, 2/. The energy integral for the linear problem is

$$E = T + \Pi^{(2)} = \mathrm{const} \tag{2.1}$$

Here T is given by the same expression (1.4), with the integration performed over the unperturbed region $\tau \cup \tau_1$, corresponding to the equilibrium state $q_j = 0, \Delta x_i = 0$; $\Pi^{(2)}$ is the first (quadratic) term in the expansion of the potential energy Π in powers of the displacements Δx_i and ξ_i. The form $\Pi^{(2)}$, if expressed in suitable notation, is precisely the second variation of the potential energy Π /1, 2/.

Let us assume that the failure of Π to have a minimum is such that there exists a field of displacements q_j^*, Δx_i^* such that the second variation of Π is negative, i.e.,

$$\Pi^{(2)} = \Pi^* < 0 \quad \text{if} \quad q_j = q_j^*, \quad \Delta x_i = \Delta x_i^* \tag{2.2}$$

The fundamental point in our estimate for the growth of the perturbations is the following representation for the derivative of W (1.3) with respect to time (/1/, p.180):

$$W^{\cdot} = 2\,(T - \Pi^{(2)}) \tag{2.3}$$

Combining (2.1) and (2.3), we get

$$W^{\cdot} = 4T - 2E \tag{2.4}$$

On the other hand, using the Cauchy inequality and the representation (1.3), we obtain an estimate

$$W^2 \leqslant \int \rho v^2\, d\tau \int \rho \xi^2\, d\tau = 2TM, \quad M \equiv \int \rho \xi^2\, d\tau \tag{2.5}$$

Using the fact that $M^{\cdot} = 2W$ and the relation (2.4), we obtain from (2.5) a differential inequality

$$d\,(M^{\cdot}/M)/dt \geqslant -4E/M \tag{2.6}$$

which, in view of (2.1), can be integrated exactly. The initial data are defined in terms of two independent vector-valued functions

$$\xi\,(\mathbf{x}, 0) = \xi^\circ(\mathbf{x}),\ \mathbf{v}\,(\mathbf{x}, 0) = \mathbf{v}^\circ(\mathbf{x}) \tag{2.7}$$

The fields $\xi^\circ(\mathbf{x})$, $\mathbf{v}^\circ(\mathbf{x})$ must satisfy the obvious conditions for an incompressible liquid, which are most simply written down by restating the initial data originally specified in terms of q_j, $q_j^{\cdot}$, Δx_i, u_i (1.6)-(1.9). The expression obtained upon integrating (2.6) is rather awkward. For our present purposes the following coarser estimate of the growth of M will suffice.

Choose the initial data (2.7) so that the value of the energy integral (2.1) is negative. To that end, using (2.2), we take $\Pi^{(2)}(0) < 0$, $T(0) < |\Pi^{(2)}(0)|$. Inequality (2.6) now implies the inequality $d\,(M^{\cdot}/M)/dt > 0$, and integration of this gives

$$M^{\cdot}/M > 2\lambda; \quad \lambda \equiv W(0)/M(0) \tag{2.8}$$

Integrating again, we obtain the inequality

$$M(t) > M(0)\exp(2\lambda t) \tag{2.9}$$

Since $W(0)$ is a bilinear form in the fields (2.7), the value of the constant λ may always be chosen to be positive. Indeed, if the choice of initial data (2.7) yields $\lambda < 0$, we need only change the sign of one of the functions $\xi^\circ(\mathbf{x})$ or $\mathbf{v}^\circ(\mathbf{x})$, leaving the other unchanged. This operation does not affect the value of E. Thus, by construction, initial data always exist implying (2.9) with $\lambda > 0$.

We have thus shown that if Condition (2.2) is satisfied, the equilibrium state of a body containing cavities filled with an ideal incompressible liquid having surface tension is unstable to a first approximation. The perturbations increase at least exponentially. The growth increment of the perturbations has a lower bound set by the quantity λ (2.8) which depends only on the initial data.

It is particularly important to determine bounds for the number λ. To that end we can usefully consider a narrower class of initial data than (2.7), containing a function $\xi^*(\mathbf{x})$ (corresponding to Δx_i^*, q_j^* (2.2), (1.8), (1.9)) and an arbitrary constant k:

$$\xi(\mathbf{x}, 0) = \xi^*(\mathbf{x}), \quad \mathbf{v}(\mathbf{x}, 0) = \xi_t(\mathbf{x}, 0) = k\xi^*(\mathbf{x}) \tag{2.10}$$

For these initial data, it follows from the definition of λ (2.8) that $\lambda = k$, and the conditions $\lambda > 0$, $E < 0$ imply the bounds

$$0 < \lambda < \Lambda \equiv \sqrt{-2\Pi^{(2)}(0)/M(0)} \tag{2.11}$$

Thus, in the case of initial data of the class (2.10) the estimate (2.9) may hold for arbitrary values of λ in the interval (2.11).

A special feature of the class (2.10) is that the corresponding λ values are the largest possible. Indeed, for perturbations with arbitrary initial data (2.7), the definition of λ (2.8), the Cauchy inequality and the condition $E < 0$ imply the upper bound $\lambda < \Lambda$.

Remark 1. Since the mathematical aspects of the existence of solutions have not been considered here, inequality (2.9) is in the nature of an a priori estimate.

Remark 2. A remarkable property of the above estimate for the growth of the perturbations is that it was derived independently of the specific form of the second variation of the potential energy $\Pi^{(2)}$. The only prerequisite for the validity of (2.9) is the existence of a perturbation with negative $\Pi^{(2)}$ (2.2) and the truth of (2.3).

Remark 3. An interesting problem in determining the largest value Λ^+ of the upper bound Λ (2.11) for all kinematically admissible fields $\xi^*(\mathbf{x})$ (2.2). Solution of this problem would make it possible to determine not only the largest values of λ but also to ascertain the actual form of those initial data (2.10) most "dangerous" in this respect. The variational problem arising here reduces to minimizing the functional $\Pi^{(2)}$ conditional on $M = 1$.

Remark 4. Proofs of the instability of the above system in various special cases, using methods of spectral theory, may be found in /3, 4/. The formulation of the problem studied in /3/ is the same as that considered here, but there the surface tension forces were not taken into account. On the other hand, surface tension was considered in /4/ but only in the case of a stationary vessel. In /3, 4/ the existence of eigenvalues that yield exponential growth of the perturbations is proved, but without supplying estimates of the growth rate.

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A TURBULENT VORTICAL DYNAMO*

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The possibility of the spontaneous appearance of rotational motion in a half-space above a plane, caused by bifurcation at some Reynolds number determining the intensity of the given sources of motion not giving rise to external force moments, is studied in the class of selfsimilar conical flows of an incompressible fluid of variable viscosity. The impossibility of spontaneous rotation is shown for the cases of constant viscosity and state of rest, and of weak sources of the basic flow. Examples of the bifurcations of the autorotation are constructed for an ascending, one-cell motion under the condition that there is no rotational friction, and for a two-cell motion with conditions of regularity on the axis and adhesion at the fixed plane. In these cases the motion is made up of an outer laminar flow, and a turbulent, high viscosity kernel near the axis. The examples quoted obviously model rotating astrophysical jets, the initiation of a whirlpool, and the onset of a firestorm above a plane under the action of a quadrupole heat source.

1. The concept of a vortical dynamo. We shall use the name "vortical dynamo" to describe

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